On higher-dimensional Teter rings via the canonical trace ideal

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This talk is based on joint work [3] with Sora Miyashita of The University of Osaka. Teter rings, introduced by W. Teter [1], are the classical class of Artinian local rings and closely related to Gorenstein properties. This notion was extended to general Noetherian local rins by T. J. Puthenpurakal [2]. In our paper [3], we introduce its graded analogue.

Let $R = \bigoplus_{i \geq 0} R_i$ be a non-Gorenstein Cohen–Macaulay graded ring, where R_0 is a field. Set $\mathfrak{m} := \bigoplus_{i>0} R_i$ and let $\omega_R = \bigoplus_{i \in \mathbf{Z}} [\omega_R]_i$ denote its graded canonical module. We define graded Teter rings as follows:

Definition. We say that R is a *Teter ring* if there exists a graded R-homomorphism $\varphi : \omega_R \to R$ such that satisfies either one of the following conditions:

(1) $\varphi(\omega_R) = \mathfrak{m}$ (this happens only when $\dim R = 0$);

(2) φ is injective and $\operatorname{embdim}\big(R/\varphi(\omega_R)\big) \leq \dim R$.

If R is a Teter ring in our sense, then $R_\mathfrak{m}$ is a Teter rings in the sense of [2]. The converse holds true if R is a domain. We study Teterness using the *canonical trace ideal* $\operatorname{tr}(\omega_R)$. To state our main result, we fix some notation. Let $r(R)$ denote the *Cohen–Macaulay type*, and a_R the *a-invariant* of R. Set

$$r_0(R) := \dim_{R_0}([\omega_R]_{-a_R}), \quad \operatorname{indeg}(\mathfrak{m}) := \min\{i \in \mathbb{Z} : [\mathfrak{m}]_i \neq 0\}.$$

We say that R is *level* if $r(R) = r_0(R)$. An element $x \in \omega_R$ is called *torsion-free* if $rx = 0$ implies $r = 0$ for all $r \in R$.

Main Theorem. *Assume that* $\dim(R) > 0$. *Consider the following conditions:*

(1) *R is level,* $r(R) \geq \operatorname{pd}(R)$*, and* $[\operatorname{tr}(\omega_R)]_{\operatorname{indeg}(\mathfrak{m})}$ *contains a non-zerodivisor;*

(2) $[\omega_R]_{-a_R}$ *contains a torsion-free element of* ω_R *(e.g., R is a domain),* $r_0(R) \geq \operatorname{pd}(R)$*, and* $[\operatorname{tr}(\omega_R)]_{\operatorname{indeg}(\mathfrak{m})}$ *contains a non-zerodivisor;*

(3) *R is a level Teter ring;*

(4) *R is a Teter ring* (*which implies* $r(R) = \operatorname{pd}(R)$).

Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *hold. Moreover, if R is standard graded, all the conditions are equivalent.*

In this talk, we explain the above result and present several applications to standard graded rings. We then discuss Teterness for numerical semigroup rings, which form the typical class of non-standard one-dimensional Cohen–Macaulay rings.

References

[1] W. Teter. Rings which are a factor of a Gorenstein ring by its socle. *Inventiones mathematicae*, 23:153–162, 1974.

[2] T. J. Puthenpurakal. Higher dimensional Teter rings. *Mediterranean Journal of Mathematics*, 22(194), 2025.

[3] S. Miyashita and T. Ozaki. On higher-dimensional Teter rings via the canonical trace ideal, in preparation.

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